On Division of Series.

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If the series $\Sigma\alpha_\rho x^\rho$ is divided by the series $\Sigma\beta_\sigma x^\sigma$, the quotient will be a series whose coefficients may be expressed in determinant form in the following manner:

$$\sum_{\rho=0}^{\rho=r}\alpha_\rho x^\rho \div \sum_{\sigma=0}^{\sigma=s}\beta_\sigma x^\sigma = \sum_{\tau=0}^{\tau=\infty}\frac{(-1)^\tau}{\beta_0^{\tau+1}}\Delta_\tau,\ \Delta_\tau = \begin{vmatrix} \alpha_0 & \beta_0 & 0 & \dots & 0 \\ \alpha_1 & \beta_1 & \beta_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_\tau & \beta_\tau & \beta_{\tau-1} & \dots & \beta_1 \end{vmatrix}.$$

This proposition may be proved either directly, by performing the indicated division, or indirectly by applying the method of indeterminate coefficients. In the first case the following formula will be found which may serve as a definition of the above determinant:

$$\Delta_{\tau+1} = \beta_1\Delta_\tau - \beta_0\Delta^\tau_{\tau+1} \tag{1}$$

where $\Delta^\tau_{\tau+1}$ denotes a determinant of the above form whose last line begins with $\alpha_{\tau+1}$, but in which the line beginning with α_τ is wanting; in the second case we assume the quotient to be $\Sigma\gamma_\tau x^\tau$, and multiplying it by the divisor find the identical equation $\Sigma\alpha_\rho x^\rho = \Sigma\Sigma\beta_\sigma\gamma_\tau x^{\sigma+\tau}$. Then changing the index τ by the formula $\sigma+\tau=\omega$ we have the equation

$$\sum_{\rho=0}^{\rho=r}\alpha_\rho x^\rho = \sum_{\omega=0}^{\omega=\infty}x^\omega\sum_{\sigma=0}^{\sigma=s}\beta_\sigma\gamma_{\omega-\sigma},$$

from which may be deduced the following two

$$\alpha_\omega = \sum_{\sigma=0}^{\sigma=s}\beta_\sigma\gamma_{\omega-\sigma},\ (r \geqq \omega \geqq \sigma);\ 0 = \sum_{\sigma=0}^{\sigma=s}\beta_\sigma\gamma_{\omega-\sigma},\ (\omega > r). \tag{2}$$

Writing these systems in the following way:

$$\begin{aligned}
\alpha_0 &= \beta_0\gamma_0 \\
\alpha_1 &= \beta_1\gamma_0 + \beta_0\gamma_1 \\
&\dots\dots\dots\dots \\
\alpha_\omega &= \beta_\omega\gamma_0 + \beta_{\omega-1}\gamma_1 + \dots + \beta_0\gamma_\omega \\
0 &= \beta_{\omega+1}\gamma_0 + \beta_\omega\gamma_1 + \dots + \beta_1\gamma_\omega + \beta_0\gamma_{\omega+1}\ \text{etc.,}
\end{aligned}$$

and remembering that for the solution with regard to γ_τ we need only the first τ equations, the following equation results:

$$\gamma_\tau \begin{vmatrix} \beta_0 & 0 & \ldots 0 \\ \beta_1 & \beta_0 & \ldots 0 \\ \ldots & \ldots & \ldots \\ \beta_{\tau-1} & \beta_{\tau-2} & \ldots 0 \\ \beta_\tau & \beta_{\tau-1} & \ldots \beta_0 \end{vmatrix} = \begin{vmatrix} \beta_0 & 0 & \ldots \alpha_0 \\ \beta_1 & \beta_0 & \ldots \alpha_1 \\ \ldots & \ldots & \ldots \\ \beta_{\tau-1} & \beta_{\tau-2} & \ldots \alpha_{\tau-1} \\ \beta_\tau & \beta_{\tau-1} & \ldots \alpha_\tau \end{vmatrix}, \text{ or } \gamma_\tau = \frac{(-1)^\tau}{\beta_0^{\tau+1}} \begin{vmatrix} \alpha_0 & \beta_0 & \ldots 0 \\ \alpha_1 & \beta_1 & \ldots 0 \\ \ldots & \ldots & \ldots \\ \alpha_{\tau-1} & \beta_{\tau-1} & \ldots \beta_0 \\ \alpha_\tau & \beta_\tau & \ldots \beta_1 \end{vmatrix}$$

as stated in the proposition. When $\omega > r$, we have of course $\alpha_\omega = 0$.

In case the two given series are finite, the quotient will be a Recurring Series, *i. e.* beginning from a certain index each determinant may be expressed as a linear function of a constant number of preceding determinants. This linear function is represented in the second formula of (2), which, after substituting the value of γ, reads as follows:

$$\sum_{\sigma=0}^{\sigma=s} (-1)^\sigma \beta_\sigma \beta_0^\sigma \Delta_{\omega-\sigma} = 0, \; (\omega > r).$$

The preceding results may be applied also to multiple series. If the dividend is a multiple sum with the general term $a_{\rho,\kappa \ldots \lambda}$, the division may be performed upon any of the indices, say the second, κ; and we shall have

$$\sum_\rho \sum_\kappa \ldots \sum_\lambda a_{\rho,\kappa \ldots \lambda} \div \sum_\sigma b_\sigma = \sum_\rho \sum_\tau \ldots \sum_\lambda \frac{(-1)^\tau}{b_0^{\tau+1}} \begin{vmatrix} a_{\rho,0 \ldots \lambda} & b_0 & 0 \\ a_{\rho,1 \ldots \lambda} & b_1 & 0 \\ a_{\rho,\tau \ldots \lambda} & b_\tau & b_1 \end{vmatrix}.$$

If the divisor is a multiple sum $\sum_\sigma \sum_\phi \ldots \sum_\chi b_{\sigma,\phi \ldots \chi}$, it may be reduced to a simple sum by putting $\sum_\phi \ldots \sum_\chi b_{\sigma,\phi \ldots \chi} = \beta_\sigma$, whereupon it admits at once of being operated upon by the formula above obtained.